What would it take to kill all the wights in the Battle of Winterfell one by one

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This paper models the Battle of Winterfell in Game of Thrones and concludes that for the Army of the Living to kill every wight one by one, each living soldier would have to kill at least 1.7 times more enemies than a wight.

In the most recent season of Game of Thrones, the TV series based on George R.R. Martin's epic fantasy series 'A Song of Ice and Fire', Winterfell is the epicenter of a battle between the alliance of living armies, including the forces of the Starks, Arryns, and Targaryens, and the army of the dead led by the White Walkers. According to several references during the previous seasons [1], the size of the living army is about 111,000 soldiers while the size of the dead army is about 100,000 soldiers. In our model for the battle we will consider every soldier within the same army to have equal killing rate (we know that this model underestimates the importance of hero characters and creatures like dragons or giants, which might leave the door open for a sequel paper!). We also know that the Army of the Dead can gain some soldiers during the battle from reanimated corpses of humans - we assume every death in the Army of the Living will account for an extra soldier for the Army of the Dead. With these assumptions we get the following pair of differential equations

$$\begin{aligned} D'(t) &= -aL(t) + bD(t) \\ L'(t) &= -bD(t) \end{aligned} \tag{1}$$

where a and b are the killing rates of a living soldier and a soldier in the dead army respectively, $L(t)$ is the number of living soldiers and $D(t)$ is the number of soldiers in the dead army in hundreds of thousands as a function of time. As initial conditions we have $L(0) = 111$ and $D(0) = 100$.

Joining the two expressions above we get the following second order linear differential equation with constant coefficients

$$-\frac{D''(t)}{b} + D'(t) + aD(t) = 0 \tag{2}$$

Writing the characteristic equation and solving for $D(t)$ we get

$$\begin{aligned} D(t) = {} & \\ & \frac{1}{\sqrt{b}\sqrt{4a+b}}\Big(-111ae^{\frac{1}{2}t\left(\sqrt{b}\sqrt{4a+b}+b\right)} + 111ae^{\frac{1}{2}t\left(b-\sqrt{b}\sqrt{4a+b}\right)} \\ & + 50be^{\frac{1}{2}t\left(\sqrt{b}\sqrt{4a+b}+b\right)} + 50\sqrt{b}\sqrt{4a+b}e^{\frac{1}{2}t\left(\sqrt{b}\sqrt{4a+b}+b\right)} \\ & - 50be^{\frac{1}{2}t\left(b-\sqrt{b}\sqrt{4a+b}\right)} + 50\sqrt{b}\sqrt{4a+b}e^{\frac{1}{2}t\left(b-\sqrt{b}\sqrt{4a+b}\right)}\Big) \end{aligned} \tag{3}$$

Now that $D(t)$ has been determined we just need to solve the equation $D(t) = 0$, and we get the combination of real positive parameters a and b that will result in the Army of the Dead being defeated. The solution for $D(t) = 0$ gives us the following condition

$$\frac{a}{b} > \frac{21100}{12321} \approx 1.71 \tag{4}$$

In other words, as long as the killing rate of the soldiers in the Army of the Living is at least 1.7 times the killing rate of the Army of the Dead the living will win!

In the end, this is a simplistic model and it was a fun excuse to explore a part of the Game of Thrones universe under a mathematical lens. The author is aware that previous rational approaches used to predict any outcomes on the show have failed miserably.

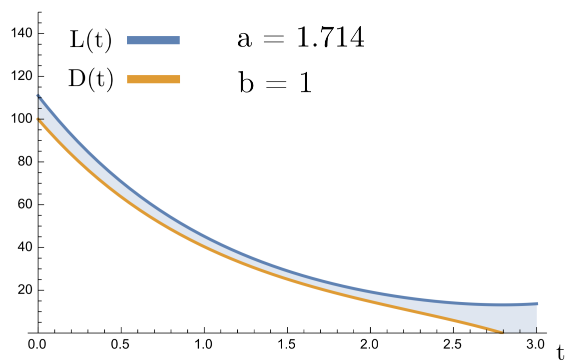

FIG. 1: Evolution of the living ($L(t)$) and dead ($D(t)$) armies with killing rates $a = 1.714$ and $b = 1$.

[1] Army Size Comparison: White Walkers Army of the Dead vs Army of Allies | Game of Thrones Season 8 (Youtube)